Upper Semi–Continuity and Directional Derivatives of Marginal Functions

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Abstract. It is known that the marginal function is upper semi-continuous if the maximand is upper semi-continuous, the feasible multifunction is upper semi-continuous in the sense of Berge and it takes compact values. In this note we give a sufficient condition to establish the upper semi-continuity of the marginal function without compactness assumption. The result applied to the directional differentiability of a marginal function with constant feasible set.

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1 Preliminaries

Let W and X be metric spaces which are called the parameter space and the decision space respectively. Given a function $f : W \times X \to R$ and a multifunction $F : W \rightrightarrows X$, define $m : W \to \overline{R} := R \cup \{+\infty\}$ by

$$m(w) := \sup_{x \in F(w)} f(w, x).$$

The resulting function m is called a marginal-function. In the first part of this note we are concerned with the upper semi-continuity of m. We recall two kinds of upper semi-continuity of F.

DEFINITION 1.1 (Berge [1,3,4]) $F : W \rightrightarrows X$ is *upper semi-continuous* at w if for any open set $V \in \mathcal{O}_X$ with $F(w) \subset V$, there exists a neighborhood U of w such that

$$F(u) \subset V, \quad \text{for all } \ u \in U.$$

DEFINITION 1.2 (Penot [12]) $F : W \rightrightarrows X$ is *graphically upper semi-continuous* at w if for any open set $\mathcal{V} \in \mathcal{O}_{W \times X}$ with $\{w\} \times F(w) \subset \mathcal{V}$, there exists a neighborhood U of w such that

$$\{u\} \times F(u) \subset \mathcal{V}, \quad \text{for all } \ u \in U.$$

Hence F is graphically upper semi-continuous at w if and only if the multifunction $u \rightrightarrows \{u\} \times F(u)$ is upper semi-continuous at w.

If one assume graphical upper semi-continuity, one gets the following upper semi-continuity result for m.

THEOREM 1.1 (Penot [12, Proposition 2.1]) *If f is upper semi-continuous at each point of $\{w\} \times F(w)$ and if F is graphically upper semi-continuous at w, then m is upper semi-continuous at w.*

Theorem 1.1 is mathematically favorable because it is simple and does not require any superfluous assumption. However, practically, it has a drawback, that is to say, the graphical upper semi-continuity is much more restrictive than the upper semi-continuity in the sense of Berge. For example a constant-valued multifunction may not be graphically upper semi-continuous.

On the other hand, when one works with the upper semi-continuity in the sense of Berge, one usually assume an additional assumption such as compactness of F to ensure the upper semi-continuity of m (see Corollary 2.1 below). In the next section, we will give a condition which establishes the upper semi-continuity of m in place of compactness of F. The result will be applied to derive the directional differentiability of marginal function of constant feasible multifunction in section 3.

There are many papers that deal with the directional derivative of marginal functions [2, 5, 6, 7, 8, 9, 10, 11, 14, 15]. In these papers, the directional differentiability was derived under the condition that the exact or approximate solution multifunctions satisfy some kind of compactness(see [2, 10, 14, 15]). In this note, we derive a similar result under the condition that the approximate solution multifunction is upper semi-continuous in the sense of Berge.

2 Upper Semi–Continuity

Given $w \in W$, define a multifunction $F_w : W \rightrightarrows X$ by

$$F_w(u) := \begin{cases} F(w) & \text{if } u = w \\ F(u) \setminus F(w) & \text{if } u \neq w. \end{cases}$$

The upper semi-continuity of F in the sense of Berge is completely characterized through the behavior of F_w (see Proposition 2.1 below). To see this fact, we present the following two lemmas.

LEMMA 2.1 *Assume $F : W \rightrightarrows X$ is upper semi-continuous at w. If $(u_n) \to w$ with $u_n \neq w$, $x_n \in F_w(u_n)$ and $(x_n) \to x$, then we have $x \in F(w)$.*

Proof. Suppose $x \notin F(w)$. Set $\mathcal{B} := \{x_n\} \cup \{x\}$. Then $\mathcal{B}$ is a closed set and $\mathcal{B} \cap F(w) = \emptyset$. If we set $V := X \setminus \mathcal{B}$, then V is open with $F(w) \subset V$. By the upper semicontinuity of F, for sufficiently large n, we have $x_n \in V = X \setminus \mathcal{B}$ a contradiction. □

LEMMA 2.2 *Assume* $F : W \rightrightarrows X$ *is upper semi-continuous at* w. *Then for all* $(u_n) \to w$ *with* $u_n \neq w$ *and for all* $x_n \in F_w(u_n)$, (x_n) *has an accumulation point.*

Proof. Let $(u_n) \to w, x_n \in F_w(u_n)$. Set $\mathcal{A} := \{x_n\}$. Suppose $\mathcal{A}$ does not have an accumulation point. Then for any $x \in X$, there exists a neighborhood $\mathcal{U}(x)$ of x such that $\mathcal{U}(x) \cap (\mathcal{A} \setminus \{x\}) = \emptyset$. Hence for each $x \in X \setminus \mathcal{A}$, we have $\mathcal{U}(x) \cap \mathcal{A} = \emptyset$, i.e. $\mathcal{U}(x) \subset X \setminus \mathcal{A}$. This means $X \setminus \mathcal{A}$ is an open set. Set $V := X \setminus \mathcal{A}$. Clearly $F(w) \subset V$. By the upper semi-continuity of F, we have $x_n \in F(u_n) \subset V = X \setminus \mathcal{A}$ for sufficiently large n. This leads to a contradiction. Hence we get that $\mathcal{A}$ has an accumulation point. □

PROPOSITION 2.1 F *is upper semi-continuous at* w *if and only if* F_w *is graphically upper semi-continuous at* w.

Proof. (Only if) Suppose not. Then there exist an open set $\mathcal{V}$ with $\{w\} \times F(w) \subset \mathcal{V}$, $(u_n) \to w$, and $x_n \in F_w(u_n)$ such that $(u_n, x_n) \notin \mathcal{V}$, which also implies $u_n \neq w$. By Lemmas 2.1 and 2.2, we may assume $(x_n) \to x \in F(w)$. Since $(u_n, x_n) \notin \mathcal{V}$, we have $(w, x) \notin \mathcal{V}$. This contradicts to $\{w\} \times F(w) \subset \mathcal{V}$. (If) Suppose the contrary. Then there exist an open set V with $F(w) \subset V$, $(u_n) \to w$, and $x_n \in F(u_n)$ such that $x_n \notin V$, which also implies $u_n \neq w$. We note that $(u_n, x_n) \in \{u_n\} \times F_w(u_n)$. Since the multifunction $u \rightrightarrows \{u\} \times F_w(u)$ is upper semicontinuous at $u = w$, by Lemmas 2.1 and 2.2, (u_n, x_n) has an accumulation point $(w, x) \in \{w\} \times F_w(w) = \{w\} \times F(w)$, which implies $x \in V$. As $X \setminus V$ is closed, we have $x \in X \setminus V$ a contradiction. □

REMARK 2.1 In the book of Bank et al, a result similar to Proposition 2.1 is indicated under the additional condition such that $F(w)$ is closed (see [3, Lemma 2.2.2]).

Now we can present the upper semi-continuity result of the marginal function with the aid of Theorem 1.1 and Proposition 2.1. It involves the following condition:

(H) the function $u \to \sup_{x \in F(w)} f(u, x)$ is upper semi-continuous at w.

THEOREM 2.1 *Let* F *be upper semi-continuous at* w *and* f *be upper semi-continuous at each point of* $\{w\} \times F(w)$. *If the condition* **(H)** *is satisfied, then* m *is upper semi-continuous at* w.

Proof. Set m_1 and m_2 as follows.

$$\begin{aligned} m_1(u) &:= \sup_{x \in F_w(u)} f(u,x), \\ m_2(u) &:= \sup_{x \in F(u) \cap F(w)} f(u,x). \end{aligned}$$

By Theorem 1.1 and Proposition 2.1, m_1 is upper semi-continuous at w. On the other hand, by the assumption **(H)**, we have

$$\limsup_{u \to w} m_2(u) \leq \limsup_{u \to w} \sup_{x \in F(w)} f(u,x) \leq \sup_{x \in F(w)} f(w,x) = m_2(w).$$

Since $m(u) = \max\{m_1(u), m_2(u)\}$, we get the desired result. □

One may think that the assumption **(H)** seems to be rather artificial. However, in the situations of classical cases, it is automatically satisfied (see Corollaries 2.1 and 2.2 below). The following example also illustrates the importance of the assumption **(H)**.

EXAMPLE 2.1 Let $W = R_+$, $X = R$, and $f(w,x) = wx$. Define $F : W \rightrightarrows X$ by

$$F(w) = \begin{cases} [0, +\infty) & \text{if } w = 0 \\ [0, \dfrac{1}{w}] & \text{if } w \neq 0. \end{cases}$$

Clearly f is continuous and F is upper semi-continuous in the sense of Berge. On the other hand, we get

$$m(w) = \begin{cases} 0 & \text{if } w = 0 \\ 1 & \text{if } w \neq 0. \end{cases}$$

which is not upper semi-continuous at $w = 0$. Indeed, the assumption **(H)** is not satisfied because

$$\sup_{x \in F(0)} f(u,x) = \begin{cases} 0 & \text{if } u = 0 \\ +\infty & \text{if } u > 0. \end{cases}$$

Now let us show that the preceding result encompasses two classical cases.

COROLLARY 2.1 (see [1, Theorem 1.4.16]) *Let F be upper semi-continuous at w and f be upper semi-continuous at each point of $\{w\} \times F(w)$. Assume that $F(w)$ is compact. Then m is upper semi-continuous at w.*

Proof. It is known that a multifunction which takes constant compact value is graphically upper semi-continuous. Thus (H) is satisfied. □

For a given upper semi-continuous function $f_0 : X \to R$, define

$$m_0(u) := \sup_{x \in F(u)} f_0(x).$$

COROLLARY 2.2 (see [3, Theorem 4.2.3]) *Let F be upper semi-continuous at w and f_0 be upper semi-continuous at each point of $\{w\} \times F(w)$. Then m_0 is upper semi-continuous at w.*

Proof. The function $u \to \sup_{x \in F(w)} f_0(x)$ is constant, hence upper semi-continuous. □

3 Directional Derivative

In this section, we treat a case in which the multifunction of the feasible constraints F is constant: for some subset C of X one has $F(w) = C$ for each $w \in W$. Hence we have

$$m(w) = \sup_{x \in C} f(w, x).$$

Throughout the sequel, the parameter space W is assumed to be a normed vector space.

3.1 Upper Derivative

Let us denote the *upper derivatives* of m and $f_x := f(\cdot, x)$ by

$$\begin{aligned} m^{\#}(w, u) &= \limsup_{(t,v) \to (0_+, u)} t^{-1}(m(w + tv) - m(w)), \\ f_x^{\#}(w, u) &= \limsup_{(t,v) \to (0_+, u)} t^{-1}(f(w + tv, x) - f(w, x)). \end{aligned}$$

Define the solution multifunction S and the approximate solution multifunction S_ε for $\varepsilon > 0$ by

$$\begin{aligned} S(w) &= \{x \in C : f(w, x) = m(w)\}, \\ S_\varepsilon(w) &= \{x \in C : f(w, x) \geq \min[m(w) - \varepsilon, 1/\varepsilon]\}. \end{aligned}$$

The following estimate is immediate [14, Lemma 3.1].

LEMMA 3.1 *Suppose $m(w_0)$ is finite and $S(w_0)$ is nonempty. Then we have*

$$\sup\{f_x^{\#}(w, u) : x \in S(w_0)\} \leq m^{\#}(w, u).$$

To show the reverse inequality, we will use the results of the preceding section. Given w_0 for which $m(w_0)$ is finite, define $q : R_+ \times W \times C \to \overline{R}$ by

$$q(t, v, x) := \begin{cases} t^{-1}(f(w_0 + tv, x) - f(w_0, x)) & \text{if } t \neq 0 \\ f_x^{\#}(w_0, v) & \text{if } t = 0. \end{cases}$$

Define also $\bar{q} : R_+ \times W \to \overline{R}$ by

$$\bar{q}(t, v) := \sup_{x \in S(w_0)} q(t, v, x).$$

THEOREM 3.1 *Let $m(w_0)$ be finite and $S(w_0)$ be nonempty. Assume that*

(A1) *q is upper semi-continuous at each point of $\{0\} \times \{u\} \times S(w_0)$,*

(A2) *for each $\varepsilon > 0$, the multifunction $(t, v) \in R_+ \times W \rightrightarrows S_{\varepsilon t}(w_0 + tv)$ is upper semi-continuous at $(0, u)$,*

(A3) *$\bar{q}$ is upper semi-continuous at $(0, u)$.*

Then we have

$$m^{\#}(w_0, u) = \sup\{f_x^{\#}(w_0, u) : x \in S(w_0)\}.$$

Proof. We should treat two cases.
Case 1: For any $\delta > 0$, there exist $t \in]0, \delta[$ and $v \in B(u, \delta)$ such that $m(w_0 + tv) = +\infty$. In this case, $m^{\#}(w_0, u) = +\infty$ and there exist $t_n \to 0^+$ and $v_n \to u$ such that $m(w_0 + t_n v_n) = +\infty$. Given $\varepsilon > 0$, take $x_n \in S_{\varepsilon t_n}(w_0 + t_n v_n)$. Then we have

$$\begin{aligned} {t_n}^{-1}((\varepsilon t_n)^{-1} - m(w_0)) &\leq {t_n}^{-1}(f(w_0 + t_n v_n, x_n) - f(w_0, x_n)) \\ &\leq \sup_{x \in S_{\varepsilon t_n}(w_0 + t_n v_n)} q(t_n, v_n, x). \end{aligned}$$

By Theorem 2.1, $(t, v) \to \sup_{x \in S_{\varepsilon t}(w_0 + tv)} q(t, v, x)$ is upper semi-continuous at $(0, u)$. If we take the upper limit of both sides, then we have,

$$\begin{aligned} +\infty &\leq \limsup_{n \to +\infty} \sup_{x \in S_{\varepsilon t_n}(w_0 + t_n v_n)} q(t_n, v_n, x) \\ &\leq \sup_{x \in S(w_0)} q(0, u, x) \\ &= \sup_{x \in S(w_0)} f_x^{\#}(w_0, u). \end{aligned}$$

Case 2: There exists $\delta > 0$ such that for any $t \in]0, \delta[$ and $v \in B(u, \delta)$, $m(w_0 + tv) < +\infty$. In this case, for any $\varepsilon > 0$, we have

$$t^{-1}(m(w_0 + tv) - m(w_0)) \leq \sup_{x \in S_{\varepsilon t}(w_0 + tv)} q(t, v, x) + \varepsilon.$$

By upper semi-continuity of $(t, v) \to \sup_{x \in S_{\varepsilon t}(w_0 + tv)} q(t, v, x)$, we have

$$m^{\#}(w_0, u) \leq \sup_{x \in S(w_0)} f_x^{\#}(w_0, u) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get the required result. □

A result similar to Theorem 3.1 was given in many literatures [2, 5, 6, 7, 8, 9, 10, 11, 14, 15]. In these studies, one pose compactness condition on $F(w_0)$ or sequentially compactness condition for the selection of F. (see, for example, [6, 14]). Our condition **(A3)** is a kind of tax to pay for the lack of compactness on F. Indeed if we assume the compactness on F, we immediately get the following corollary.

COROLLARY 3.1 *Under the conditions of Theorem 3.1, we assume that $S(w_0)$ is compact in place of* **(A3)**. *Then*

$$m^{\#}(w_0,u)=\sup\{f_x^{\#}(w_0,u):x\in S(w_0)\}.$$

Proof. From **(A1)** and compactnes of $S(w_0)$, by Corollary 2.1, it is assured that the marginal function $\bar{q}(t,v):=\sup_{x\in S(w_0)}q(t,v,x)$ is upper semi-continuous at $(0,u)$. Hence **(A3)** holds. □

The condition **(A1)** is equivalent to the following condition (see [14]).

$$(\mathbf{A1'})\quad f_x^{\#}(w_0,u)=\limsup_{(t,v,x')\to(0^+,u,x)} t^{-1}(f(w_0+tv,x')-f(w_0,x'))\quad\text{for each}\quad x\in S(w_0).$$

PROPOSITION 3.1 **(A1)** *and* ($\mathbf{A1'}$) *are equivalent.*

Proof. It is clear that **(A1)** implies ($\mathbf{A1'}$). Contrary, assume ($\mathbf{A1'}$) holds. Since for $\varepsilon>0$,

$$\sup_{\substack{0\le t<\varepsilon\\ v\in B(u,\varepsilon)\\ x'\in B(x,\varepsilon)}} q(t,v,x')=\max\{\sup_{\substack{v\in B(u,\varepsilon)\\ x'\in B(x,\varepsilon)}} f_{x'}^{\#}(w_0,v),\ \sup_{\substack{0<t<\varepsilon\\ v\in B(u,\varepsilon)\\ x'\in B(x,\varepsilon)}} t^{-1}(f(w_0+tv,x')-f(w_0,x'))\}$$

and

$$\lim_{\varepsilon\to0_+}\sup_{\substack{0<t<\varepsilon\\ v\in B(u,\varepsilon)\\ x'\in B(x,\varepsilon)}} t^{-1}(f(w_0+tv,x')-f(w_0,x'))\}=f_x^{\#}(w_0,u),$$

it is enough to show $(v,x')\to f_{x'}^{\#}(w_0,v)$ is upper semi-continuous at (u,x). Take $(u_n,x_n)\to(u,x)$ arbitrary. We first show the case in which $f_{x_n}^{\#}(w_0,u_n)=+\infty$ for infinitely many n. In this case for each n, by definition of upper limit, there exist $t_n>0$ and v_n such that $t_n+d(u_n,v_n)\le 1/n$ and

$$n\le {t_n}^{-1}(f(w_0+t_nv_n,x_n)-f(w_0,x_n)).$$

Upon taking upper limits of both sides, we derive

$$\begin{aligned}
+\infty &\le \limsup_{n\to+\infty} {t_n}^{-1}(f(w_0+t_nv_n,x_n)-f(w_0,x_n))\\
&\le \limsup_{(t,v,x')\to(0^+,u,x)} t^{-1}(f(w_0+tv,x')-f(w_0,x'))\\
&= f_x^{\#}(w_0,u),
\end{aligned}$$

where the last equality is assured by ($\mathbf{A1'}$).
Assume that $f_{x_n}^{\#}(w_0,u_n)=+\infty$ for at most finite n. Then, by definition of upper limit, we can take $t_n>0$ and v_n such that $t_n+d(u_n,v_n)\le 1/n$ and

$$f_{x_n}^{\#}(w_0,u_n)-1/n\le {t_n}^{-1}(f(w_0+t_nv_n,x_n)-f(w_0,x_n)).$$

Upon taking upper limits of both sides, we derive

$$\begin{aligned}\limsup_{n\to+\infty} f^{\#}_{x_n}(w_0,u_n) &\le \limsup_{n\to+\infty} t_n{}^{-1}(f(w_0+t_nv_n,x_n)-f(w_0,x_n))\\ &\le \limsup_{(t,v,x')\to(0^+,u,x)} t^{-1}(f(w_0+tv,x')-f(w_0,x'))\\ &= f^{\#}_x(w_0,u).\end{aligned}$$

This completes the proof. □

3.2 Radial Derivative

In this part, we treat the (upper) *radial derivative* of m and f_x defined as :

$$\begin{aligned} m'_r(w,u) &= \limsup_{t\to0_+} t^{-1}(m(w+tu)-m(w)),\\ f'_x(w,u) &= \limsup_{t\to0_+} t^{-1}(f(w+tu,x)-f(w,x)).\end{aligned}$$

We will give the counter part of Theorem 3.1 for the estimation of $m'_r(w,u)$. Given w_0 for which $m(w_0)$ is finite, define $q_r : R_+\times C\to\overline{R}$ by

$$q_r(t,x) := \begin{cases} t^{-1}(f(w_0+tu,x)-f(w_0,x)) & \text{if } t\neq 0\\ f'_x(w_0,u) & \text{if } t=0.\end{cases}$$

Define also $\bar{q_r} : R_+\to\overline{R}$ by

$$\bar{q_r}(t) := \sup_{x\in S(w_0)} q_r(t,x).$$

The following result is proven in the same manner as Theorem 3.1.

THEOREM 3.2 *Let $m(w_0)$ be finite and $S(w_0)$ be nonempty. Assume that*

(B1) *q_r is upper semi-continuous at each point of $\{0\}\times S(w_0)$,*

(B2) *for each $\varepsilon>0$, the multifunction $t\in R_+\rightrightarrows S_{\varepsilon t}(w_0+tu)$ is upper semi-continuous at 0,*

(B3) *$\bar{q_r}$ is upper semi-continuous at 0.*

Then we have

$$m'_r(w_0,u)=\sup\{f'_x(w_0,u) : x\in S(w_0)\}.$$

The condition **(B1)** is equivalent to the following condition.

$$\textbf{(B1}'\textbf{)}\quad f'_x(w_0,u) = \limsup_{(t,x')\to(0^+,x)} t^{-1}(f(w_0+tu,x')-f(w_0,x')) \quad \text{for each} \quad x\in S(w_0).$$

The proof of the following proposition is as same as that of Proposition 3.1, so we omit the proof.

PROPOSITION 3.2 **(B1)** *and* (**B1**$'$) *are equivalent.*

When we work with radial derivatives, we can give rather natural condition which assure (**B1**) and (**B3**) simultaneously.

DEFINITION 3.1 ([9, 10]) f is said to be *equi-differentiable* at w_0 in the direction u provided that there exist $\eta > 0$ and $\varepsilon : R_+ \setminus \{0\} \to R_+$ with $\lim_{t\to 0} \varepsilon(t) = 0$ such that

$$|t^{-1}(f(w_0 + tu, x) - f(w_0, x)) - f'_x(w_0, u)| \le \varepsilon(t), \tag{1}$$

for all $t > 0$ and $x \in S_\eta(w_0)$.

REMARK 3.1 If f is equi-differentiable at w_0 in the direction u, then the limit

$$\lim_{t\to 0^+} t^{-1}(f(w + tu, x) - f(w, x))$$

exists for each $x \in S_\eta(w_0)$ and equal to $f'_x(w_0, u)$. The equi-differentiability also asserts that the above limit converges uniformly in $x \in S_\eta(w_0)$.

PROPOSITION 3.3 *Assume that f is equi-differentiable at w_0 in the direction u , then* (**B3**) *holds.*

Proof. Since f is equi-differentiable, we get

$$\begin{aligned} q_r(t, x) &= t^{-1}(f(w_0 + tu, x) - f(w_0, x)) \\ &\le f'_x(w_0, u) + \varepsilon(t) \\ &= q_r(0, x) + \varepsilon(t). \end{aligned}$$

for all $t > 0$ and $x \in S(w_0)$. By taking supremum in $x \in S(w_0)$, we have $\bar{q}_r(t) \le \bar{q}_r(0) + \varepsilon(t)$ for all $t > 0$. Hence we have

$$\limsup_{t\to 0} \bar{q}_r(t) \le \bar{q}_r(0).$$

This completes the proof. □

PROPOSITION 3.4 *Let $x \to f(w, x)$ be continuous for each $w \in W$. Assume that f is equi-differentiable at w_0 in the direction u , then* (**B1**) *holds.*

Proof. We first show that

$$\lim_{x'\to x} f'_{x'}(w_0, u) = f'_x(w_0, u), \tag{2}$$

holds for all $x \in S(w_0)$. Let $x \in S(w_0)$ be fixed. By the continuity of f, there exists $r > 0$ such that $B(x, r) \subset S_\eta(w_0)$, where $B(x, r)$ denotes the open ball centered at x with radius r. Given $\varepsilon > 0$, from (1), there exists $\delta > 0$ such that

$$|t^{-1}(f(w_0 + tu, x') - f(w_0, x')) - f'_{x'}(w_0, u)| \leq \varepsilon,$$

for all $t \in]0, \delta[$ and $x' \in B(x, r)$. We note that δ does not depend on x'. Let $t_0 \in]0, \delta[$ be fixed. then, by the continuity of $x \to f(w, x)$, there exists $\rho > 0$ such that for all $x' \in B(x, \rho)$

$$|(f(w_0 + t_0 u, x') - f(w_0, x')) - (f(w_0 + t_0 u, x) - f(w_0, x))| \leq \varepsilon t_0.$$

We may assume $\rho \leq r$. Hence, if $x' \in B(x, \rho)$, we have

$$\begin{aligned} &|f'_{x'}(w_0, u) - f'_x(w_0, u)| \\ &\quad\leq |t_0^{-1}(f(w_0 + t_0 u, x') - f(w_0, x')) - f'_{x'}(w_0, u)| \\ &\quad+ |t_0^{-1}(f(w_0 + t_0 u, x) - f(w_0, x)) - f'_x(w_0, u)| \\ &\quad+ t_0^{-1}|(f(w_0 + t_0 u, x') - f(w_0, x')) - (f(w_0 + t_0 u, x) - f(w_0, x))| \\ &\quad\leq 3\varepsilon. \end{aligned}$$

Once we get (2), we can easily deduce that

$$\begin{aligned} &\limsup_{(t,x') \to (0^+, x)} t^{-1}(f(w_0 + tu, x') - f(w_0, x')) \\ &\qquad\leq \limsup_{(t,x') \to (0^+, x)} (f'_{x'}(w_0, u) + \varepsilon(t)) \\ &\qquad= f'_x(w_0, u). \end{aligned}$$

By Proposition 3.2, (**B1**) holds. □

REMARK 3.2 If f is equi-differentiable, it is easily verified that

$$\sup\{f'_x(w_0, u) : x \in S(w_0)\} \leq \liminf_{t \to 0^+} t^{-1}(m(w + tu) - m(w)).$$

Hence we know the limit

$$\lim_{t \to 0^+} t^{-1}(m(w + tu) - m(w))$$

exists and equal to $\sup\{f'_x(w_0, u) : x \in S(w_0)\}$.

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