A SHORT PROOF OF THE REPRESENTATION FORMURA OF $f(x+d)-f(x)$ VIA APPROXIMATE DIRECTIONAL DERIVATIVE FOR CONVEX FUNCTION

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Theory of *minimax theorems* has been proved to be useful in the area of mathematical programming. Using minimax theorems, several important results in this area can be shown. The aim of this note is to give a short proof of the representation formula of $f(x+d)-f(x)$ via approximate directional derivative for a convex function f by using minimax theorem. The formula was given by Lrmarèchal and Zowe[3].

THEOREM. *Let f be a convex function on $\mathbb{R}^n$. Then for each x and d in $\mathbb{R}^n$ one has*

$$f(x+d)-f(x)=\max\{f'_\varepsilon(x;d)-\varepsilon : \varepsilon \geq 0\}. \tag{1}$$

Proof. First of all, we recall the definition of the approximate directional derivative $f'_\varepsilon(x;d)$, which is defined by the infimum of the approximate difference quotient over all positive number λ:

$$f'_\varepsilon(x;d) := \inf_{\lambda>0}[f(x+\lambda d)-f(x)+\varepsilon]/\lambda.$$

Hence the right hand side of the equation (1) is considered as the following max-min problem.

$$\begin{aligned} & \sup_{\varepsilon\geq 0}\inf_{\lambda>0}\{[f(x+\lambda d)-f(x)+\varepsilon]/\lambda-\varepsilon\} \\ = \; & \sup_{\varepsilon\geq 0}\inf_{\mu>0}\{[f(x+(1/\mu)d)-f(x)+\varepsilon]\mu-\varepsilon\}. \end{aligned} \tag{2}$$

If we apply minimax theorem to (2), the we have

$$\begin{aligned} & \sup_{\varepsilon\geq 0}\inf_{\mu>0}\{[f(x+(1/\mu)d)-f(x)+\varepsilon]\mu-\varepsilon\} \\ = \; & \inf_{\mu>0}\sup_{\varepsilon\geq 0}\{[f(x+(1/\mu)d)-f(x)+\varepsilon]\mu-\varepsilon\} \\ = \; & \inf_{\mu>0}\sup_{\varepsilon\geq 0}\{[f(x+(1/\mu)d)-f(x)]\mu+\varepsilon(\mu-1)\} \\ = \; & \{[f(x+(1/\mu)d)-f(x)]\mu : 0<\mu\leq 1\} \\ = \; & f(x+d)-f(x). \end{aligned}$$

In the last equation, we use the monotone decreasing property of the function $\mu \to [[f(x+(1/\mu)d)-f(x)]\mu]$

Q.E.D.

Appendix. We recall the minimax theorem which we use in the proof. The following result is taken from Gol'shtein's book[1].

THEOREM A. Let $F(\varepsilon, \mu)$ be concave in $\varepsilon \in E$ and convex in $\mu \in M$, let E and M be convex, let the set $M^* = \{\mu : \sup_{\varepsilon \in E} F(\varepsilon, \mu) = \inf_{\mu \in M} \sup_{\varepsilon \in E} F(\varepsilon, \mu)\}$ be nonempty and bounded, let the function $F(\varepsilon, \mu)$ be continuous in μ variable, and the set M be closed. Then

$$\sup_{\varepsilon \in E} \inf_{\mu \in M} F(\varepsilon, \mu) = \inf_{\mu \in M} \sup_{\varepsilon \in E} F(\varepsilon, \mu).$$

We set $E = [0, +\infty), M = [0, 1]$ and

$$\begin{aligned} F(\varepsilon, \mu) &= [f(x + (1/\mu)d) - f(x) + \varepsilon]\mu - \varepsilon, \quad \text{if} \quad \mu > 0, \\ F(\varepsilon, 0) &= f_\infty(d) - \varepsilon, \end{aligned}$$

where f_∞ denotes the recession function of f. In this case $M^* = \{1\}$, thus all the assumption of Theorem A is satisfied and the theorem is applicable. We should also note that $\inf_{\mu \in M} F(\varepsilon, \mu) = f'_\varepsilon(x; d)$, see [2].

References

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[3] C. LEMARÉCHAL AND J. ZOWE, Some remarks on the construction of higher order algorithms in convex optimization, Appl. Math. Optim., 10, (1983), pp.51–68.